

# CONDITIONALLY POSITIVE DEFINITE KERNELS IN HILBERT $C^*$ -MODULES

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**ABSTRACT.** We investigate the notion of conditionally positive definite in the context of Hilbert  $C^*$ -modules and present a characterization of the conditionally positive definiteness in terms of the usual positive definiteness. We give a Kolmogorov type representation of conditionally positive definite kernels in Hilbert  $C^*$ -modules. As a consequence, we show that a  $C^*$ -metric space  $(S, d)$  is  $C^*$ -isometric to a subset of a Hilbert  $C^*$ -module if and only if  $K(s, t) = -d(s, t)^2$  is a conditionally positive definite kernel. We also present a characterization of the order  $K' \leq K$  between conditionally positive definite kernels.

## 1. INTRODUCTION

Let  $\mathbb{B}(\mathcal{H})$  denote the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ . According to the Gelfand–Naimark–Segal theorem, every  $C^*$ -algebra can be regarded as a  $C^*$ -subalgebra of  $\mathbb{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . An operator  $A \in \mathbb{B}(\mathcal{H})$  is called *positive* if  $\langle Ax, x \rangle \geq 0$  for all  $x \in \mathcal{H}$ , and we then write  $A \geq 0$ . In the case when the dimension of  $\mathcal{H}$  is finite, that is when we deal with matrices, it is custom to use the terminology ‘*positive semi-definite*’. To unify our approach, we however use the word ‘positive’ instead of ‘positive semi-definite’.

A matrix  $A = [a_{ij}]$  in  $\mathbb{M}_n(\mathcal{A})$ , the  $C^*$ -algebra of  $n \times n$  matrices with entries in  $\mathcal{A}$ , is positive if and only if  $\sum_{i,j=1}^n a_i^* a_{ij} a_j \geq 0$  for all  $a_1, \dots, a_n \in \mathcal{A}$ . It follows from [18, Lemma IV.3.2] that a matrix in  $\mathbb{M}_n(\mathcal{A})$  is positive if and only if it is a sum of matrices of the form  $[c_i^* c_j]$  for some  $c_1, \dots, c_n \in \mathcal{A}$ .

The notion of *semi-inner product  $C^*$ -module* (Hilbert  $C^*$ -module, resp.) is a natural generalization of that of semi-inner produce space (Hilbert space, resp.) arising by replacing the field of scalars  $\mathbb{C}$  by a  $C^*$ -algebra. Recall that if  $(\mathcal{E}, \langle \cdot, \cdot \rangle)$  is a Hilbert  $C^*$ -module over a  $C^*$ -algebra  $\mathcal{A}$ , then for every  $x \in \mathcal{E}$  the norm

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on  $\mathcal{E}$  is given by  $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$  and the “absolute-value norm” is defined by  $|x| = \langle x, x \rangle^{\frac{1}{2}}$  as a positive element of  $\mathcal{A}$ . A map  $T : \mathcal{E} \longrightarrow \mathcal{F}$  between Hilbert  $C^*$ -modules is adjointable if there is a map  $T^* : \mathcal{F} \longrightarrow \mathcal{E}$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for all  $x \in \mathcal{E}$  and  $y \in \mathcal{F}$ . Then  $T$  (and  $T^*$ ) are bounded  $A$ -linear maps by the uniform boundedness theorem. The set of all adjointable maps  $T : \mathcal{E} \longrightarrow \mathcal{F}$  is denoted by  $\mathcal{L}(\mathcal{E}, \mathcal{F})$ . It is known that  $\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})$  endowed with the operator norm is a unital  $C^*$ -algebra. A Hilbert  $C^*$ -module  $\mathcal{E}$  is called self-dual if for every bounded  $A$ -linear map  $f : \mathcal{E} \rightarrow \mathcal{A}$  there exist some  $x_0 \in \mathcal{E}$  such that  $f(x) = \langle x_0, x \rangle$  for all  $x \in \mathcal{E}$ , see [12, §2.5]. A  $C^*$ -metric on a set  $S$  with values in a  $C^*$ -algebra  $\mathcal{A}$  is a map  $d$  from  $S \times S$  into the cone  $\mathcal{A}_+$  of positive elements of  $A$  satisfying the same axioms as those of the usual metric when we consider the usual order  $\leq$  on the real space of self-adjoint elements induced by the positive cone  $\mathcal{A}_+$ . See [11] for some examples of  $C^*$ -metrics. By a “ $C^*$ -isometry”  $V$  from a  $C^*$ -metric space  $S$  into a Hilbert  $C^*$ -module  $\mathcal{E}$  we mean one satisfying  $d(s, t) = |V(s) - V(t)|$ .

The Cauchy–Schwarz inequality (see also [2]) for  $x, y$  in a semi-inner product  $C^*$ -module  $\mathcal{E}$  asserts that

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle. \quad (1.1)$$

The reader is referred to [9, 10, 12] for more information on Hilbert  $C^*$ -modules.

It seems that positive definite kernels are first examined in 1904 by Hilbert, and conditionally positive definite kernels by Schoenberg around 1940 in a series of papers. Schoenberg also use conditionally positive definite kernels to embed a metric space into a Hilbert space. The (Kolmogorov) representation of positive definite kernels was first established by Kolmogorov in the scalar theory. The reader may consult [4, pages 84-85] for a complete historical view. The Kolmogorov decomposition of positive definite kernels in context of Hilbert  $C^*$ -modules was given by Murphy [13]. This decomposition asserts that any positive definite kernel  $L : S \times S \rightarrow \mathcal{L}(\mathcal{E})$  for a given Hilbert  $C^*$ -module  $\mathcal{E}$  is of the form  $L(s, t) = V(s)^*V(t)$ , where  $V$  is a map from  $S$  into  $\mathcal{L}(\mathcal{E}, \mathcal{F})$  for some Hilbert  $C^*$ -module  $\mathcal{F}$ .

In this paper, we first state the notion of conditionally (called also almost) positive definite kernel in the context of Hilbert  $C^*$ -modules as a generalization of that in the scalar theory (cf. [4, 6, 7]) and investigate some of its basic properties. Our investigation relies on the construction of Kolmogorov decomposition

given by Murphy [13]. Giving a characterization of conditionally positive definite kernels in Hilbert  $C^*$ -modules, we show that a  $C^*$ -metric space  $(S, d)$  is  $C^*$ -isometric to a subset of a Hilbert  $C^*$ -module if and only if  $K(s, t) = -d(s, t)^2$  is a conditionally positive definite kernel. We also present a characterization of the majorization  $K' \leq K$  between conditionally positive definite kernels. Among other things, we present a Cauchy–Schwarz inequality for positive definite kernels with values in Hilbert  $C^*$ -modules.

Throughout the paper,  $S$  stands for a non-empty set and  $\mathcal{A}, \mathcal{B}$  denote  $C^*$ -algebras. Hilbert  $C^*$ -modules are denoted by  $\mathcal{E}, \mathcal{F}$ . For the sake of convenience, we usually use the letter  $L$  for positive definite kernels and  $K$  for conditionally positive definite kernels.

## 2. CONDITIONALLY POSITIVE DEFINITE KERNELS AND CONDITIONALLY POSITIVE MATRICES

We model some techniques of the scalar theory of conditionally positive definite kernels to the context of Hilbert  $C^*$ -modules. We start our work with the following definition.

**Definition 2.1.** A hermitian matrix  $A = [a_{ij}] \in \mathbb{M}_n(\mathcal{A})$  ( $n \geq 2$ ) is called *conditionally positive* if  $\sum_{i,j=1}^n a_i^* a_{ij} a_j \geq 0$  whenever  $\sum_{i=1}^n a_i = 0$ .

Evidently, conditionally positive matrices form a positive cone containing positive matrices as well as matrices of the form  $[c_i + c_j^*]$  for  $c_1, \dots, c_n \in \mathcal{A}$ . This notion is closely related to some significant classes of real functions. For example, if  $f : (0, \infty) \rightarrow (0, \infty)$  is an operator concave function, then all Loewner matrices associated with  $f$  are conditionally positive; see [5]. This notion has several application in harmonic analysis, physics and probability theory; see [4] and references therein. There are other generalizations of this notion as well; see [8, §8].

By a kernel we mean any map on  $S \times S$  into a  $C^*$ -algebra for some set  $S$ . For any kernel  $L : S \times S \rightarrow \mathcal{A}$  one can define the kernel  $L^* : S \times S \rightarrow \mathcal{A}$  by  $L^*(s, t) = L(t, s)^*$ . A kernel  $L$  with values in a  $C^*$ -algebra is called hermitian if  $L^* = L$ .

**Definition 2.2.** A hermitian map  $L : S \times S \rightarrow \mathcal{A}$  is called a *positive definite kernel* if

$$\sum_{i,j=1}^n a_i^* L(s_i, s_j) a_j \geq 0. \quad (2.1)$$

for every positive integer  $n$ , every  $s_1, \dots, s_n \in S$  and every  $a_1, \dots, a_n \in \mathcal{A}$ .

If  $L$  is a kernel with self-adjoint values, then (2.1) holds if  $\sum_{i,j=1}^n a_i L(s_i, s_j) a_j \geq 0$  for any self-adjoint  $a_1, \dots, a_n \in \mathcal{A}$ . This easily follows by utilizing the Cartesian decomposition of any  $a_i$  into its real and imaginary parts.

**Definition 2.3.** If a hermitian kernel  $K : S \times S \rightarrow \mathcal{A}$  satisfies

$$\sum_{i,j=1}^n a_i^* K(s_i, s_j) a_j \geq 0. \quad (2.2)$$

for all positive integers  $n \geq 2$ , all  $s_1, \dots, s_n \in S$  and all  $a_1, \dots, a_n \in \mathcal{A}$  subject to the condition  $\sum_{i=1}^n a_i = 0$ , then it is called *conditionally positive definite*.

It follows from inequalities (2.1) and (2.2) that the conditionally positive definiteness of  $K$  and the positive definiteness of  $L$  are equivalent to those of the matrices  $[L(s_i, s_j)]$  and  $[K(s_i, s_j)]$  in  $\mathbb{M}_n(\mathcal{A})$  for all  $s_1, \dots, s_n \in S$ , respectively.

Clearly the set of all positive definite kernels and the set of all conditionally positive definite kernels constitute positive cones. It is easy to check that any the so-called *Gram kernel*  $L(s, t) = g(s)^* g(t)$  and any kernel of the form  $K(s, t) = g(s) + g(t)^*$ , where  $g : S \rightarrow \mathcal{A}$  is a map, are positive definite and conditionally positive definite, respectively. It immediately follows from the definition that if  $L$  is a positive definite kernel, then  $L(s, s) \geq 0$  and  $L(t, s) = L(s, t)^*$  for all  $s, t \in S$ .

The next theorem is known in the literature for scalar kernels; cf. see [4]. It was first proved in a special case by Schoenberg [17]. We however states its proof in the noncommutative setting of  $C^*$ -algebras to show that how the conditionally positive definiteness differs from the positive definiteness.

**Theorem 2.4.** *Let  $K : S \times S \rightarrow \mathcal{A}$  be a hermitian kernel and  $s_0 \in S$ . Then  $K$  is conditionally positive definite if and only if the kernel  $L : S \times S \rightarrow \mathcal{A}$  defined by*

$$L(s, t) := \frac{1}{2} [K(s, t) - K(s, s_0) - K(s_0, t) + K(s_0, s_0)]$$

*is a positive definite kernel.*

*Proof.* Let  $K$  be conditionally positive definite,  $s_1, \dots, s_n \in S$  and  $a_1, \dots, a_n \in A$ . Set  $a_{n+1} := -\sum_{i=1}^n a_i$  and  $s_{n+1} := s_0$ . Then

$$\begin{aligned}
 0 &\leq \frac{1}{2} \sum_{i,j=1}^{n+1} a_i^* K(s_i, s_j) a_j \\
 &= \frac{1}{2} \sum_{i,j=1}^n a_i^* K(s_i, s_j) a_j + \frac{1}{2} \sum_{i=1}^n a_i^* K(s_i, s_0) a_{n+1} \\
 &\quad + \frac{1}{2} \sum_{j=1}^n a_{n+1}^* K(s_0, s_j) a_j + \frac{1}{2} a_{n+1}^* K(s_0, s_0) a_{n+1} \\
 &= \frac{1}{2} \sum_{i,j=1}^n a_i^* [K(s_i, s_j) - K(s_i, s_0) - K(s_0, s_j) + K(s_0, s_0)] a_j \\
 &= \sum_{i,j=1}^n a_i^* L(s_i, s_j) a_j
 \end{aligned}$$

whence we conclude that  $L$  is positive definite.

Conversely, let  $L$  be positive definite,  $n \geq 2$ ,  $s_1, \dots, s_n \in S$  and  $a_1, \dots, a_n \in A$  with  $\sum_{i=1}^n a_i = 0$ . Then

$$\begin{aligned}
 0 &\leq 2 \sum_{i,j=1}^n a_i^* L(s_i, s_j) a_j \\
 &= \sum_{i,j=1}^n a_i^* K(s_i, s_j) a_j - \left( \sum_{i=1}^n a_i^* K(s_i, s_0) \right) \left( \sum_{j=1}^n a_j \right) \\
 &\quad - \left( \sum_{i=1}^n a_i^* \right) \left( \sum_{j=1}^n K(s_0, s_j) a_j \right) + \left( \sum_{i=1}^n a_i^* \right) K(s_0, s_0) \left( \sum_{j=1}^n a_j \right) \\
 &= \sum_{i,j=1}^n a_i^* K(s_i, s_j) a_j.
 \end{aligned}$$

Hence  $K$  is conditionally positive definite.  $\square$

*Remark 2.5.* Under the notation as in Theorem 2.4, the positive definite kernel  $L$  is identically zero if and only if there is a function  $h : S \rightarrow \mathcal{A}$  such that  $K(s, t) = h(s) + h(t)^*$ . The sufficiency is clear. To prove the necessity, it is enough to put  $h(s) := K(s, s_0) - \frac{1}{2}K(s_0, s_0)$ .

As a consequence of Theorem 2.4, we state the following assertion for matrices.

**Corollary 2.6.** *A matrix  $[a_{ij}]$  in  $\mathbb{M}_n(\mathcal{A})$  is conditionally positive if and only if the matrix  $[a_{ij} - a_{im} - a_{mj} + a_{mm}]$  is positive for some  $1 \leq m \leq n$ .*

**Corollary 2.7.** *A self-adjoint  $2 \times 2$  block matrix  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$  of operators in  $\mathbb{B}(\mathcal{H})$  is conditionally positive if and only if  $A + C \geq B + B^*$ .*

*Proof.* Employing Corollary 2.6 with  $m = 2$ , we see that  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$  is conditionally positive if and only if  $\begin{bmatrix} A + C - B - B^* & 0 \\ 0 & 0 \end{bmatrix}$  is positive.  $\square$

*Remark 2.8.* A linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  between  $C^*$ -algebras is said to be positive if  $\Phi(A) \geq 0$ , whenever  $A \geq 0$ . A linear map  $\Phi$  is called  $n$ -positive if the map  $\Phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$  defined by  $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$  is positive. A map  $\Phi$  is said to be *completely positive* if it is  $n$ -positive for every  $n \in \mathbb{N}$ .

Given a linear map  $\Phi$  between  $C^*$ -algebras, Corollary 2.7 ensures that if  $\Phi$  is positive, then the corresponding map  $\Phi_2$  preserves the conditional positivity. The converse can be seen to be true by considering the conditionally positive matrix  $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ . So it seems that to give a definition of completely conditionally positive map in the sense that it takes any conditionally positive matrix to a conditionally positive one needs some care. In this direction, one may say a map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  between  $C^*$ -algebras to be *completely conditionally positive* if  $\sum_{i,j=1}^n a_i^* \Phi(b_i^* b_j) a_j \geq 0$  for all  $n \geq 2$ ,  $b_1, \dots, b_n \in \mathcal{A}$  and  $a_1, \dots, a_n \in \mathcal{A}$  with  $\sum_{i=1}^n a_i b_i = 0$ . Such maps have interesting properties in studying some types of semigroups; see [3].

From Corollary 2.7 applied to the conditionally positive  $2 \times 2$  matrix

$$\begin{bmatrix} K(s, s) & K(s, t) \\ K(t, s) & K(t, t) \end{bmatrix}$$

we derive that if  $K$  is a conditionally positive definite kernel, then

$$2\operatorname{Re}K(s, t) \leq K(s, s) + K(t, t)$$

for all  $s, t \in S$ .

To achieve a Cauchy–Schwarz inequality for conditionally positive definite kernels we need the following lemma.

**Lemma 2.9.** [10, Lemma 5.2] *Let  $\mathcal{H}$  be a Hilbert space and  $T, P, Q \in \mathbb{B}(\mathcal{H})$  with  $P \geq 0$  and  $Q \geq 0$ . If  $\begin{bmatrix} P & T \\ T^* & Q \end{bmatrix} \geq 0$  in  $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ , then  $TT^* \leq \|Q\|P$ .*

Now, for a positive definite kernel  $L$ , it follows from the positivity of the matrix

$$\begin{bmatrix} L(s, s) & L(s, t) \\ L(t, s) & L(t, t) \end{bmatrix}$$

and Lemma 2.9 that the following Cauchy–Schwarz inequality holds

$$L(s, t)L(t, s) \leq \|L(t, t)\|L(s, s).$$

This inequality is a generalization of [7, Theorem 1.14] to positive definite kernels with values in  $C^*$ -algebras. We summarize the above facts as a proposition.

**Proposition 2.10** (Cauchy–Schwarz inequality). *Let  $S$  be a set.*

- (i) *Let  $K$  be a conditionally positive definite kernel on  $S$  with values in a  $C^*$ -algebra. Then  $2\operatorname{Re}K(s, t) \leq K(s, s) + K(t, t)$  for all  $s, t \in S$ .*
- (ii) *Let  $L$  be a positive definite kernel on  $S$  with values in a  $C^*$ -algebra. Then  $L(s, t)L(t, s) \leq \|L(t, t)\|L(s, s)$  for all  $s, t \in S$ .*

*Remark 2.11.* Let  $\mathcal{A}$  be a unital commutative  $C^*$ -algebra. By the Gelfand theorem it is of the form  $C(\Omega)$  for some compact Hausdorff space  $\Omega$ . It is known that the Schur product  $A \circ B = [a_{ij}b_{ij}]$  of two positive matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  is positive. Hence the product  $K_1 \circ K_2 : S \times S \rightarrow \mathcal{A}$  of two positive definite kernels defined by  $(K_1 \circ K_2)(s, t) = K_1(s, t)K_2(s, t)$  is again positive definite. This property is, however, not true for conditionally positive definite kernels. It is enough to consider  $S = \{1, 2\}$  and conditionally positive matrices  $A = B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$  and use Corollary 2.7.

### 3. KOLMOGOROV DECOMPOSITION OF CONDITIONALLY POSITIVE DEFINITE KERNELS

Murphy [13] established a Kolmogorov decomposition of positive definite kernels in context of Hilbert  $C^*$ -modules inspired by the scalar version in [7]. Utilizing his construction, we establish the Kolmogorov decomposition of conditionally positive definite kernels in the setting of Hilbert  $C^*$ -modules. Such constructions can be found in some types of Steispring theorems in various settings [1, 15].

**Theorem 3.1** (Kolmogorov decomposition). *Let  $K$  be a conditionally positive definite kernel on a set  $S$  into the  $C^*$ -algebra  $\mathcal{L}(\mathcal{E})$  for some Hilbert  $\mathcal{A}$ -module  $\mathcal{E}$ .*

Then there exist a Hilbert  $C^*$ -module  $\mathcal{F}$  over  $\mathcal{A}$  and a mapping  $V : S \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{F})$  such that

$$K(s, t) = 2V(s)^*V(t) - V(s)^*V(s) - V(t)^*V(t) - h(s) - h(t)^*, \quad (3.1)$$

where  $h : S \rightarrow \mathcal{L}(\mathcal{A})$  is a certain map.

*Proof.* Let us fix  $s_0 \in S$  and set

$$L(s, t) = \frac{1}{2} (K(s, t) - K(s, s_0) - K(s_0, t) + K(s_0, s_0)). \quad (3.2)$$

Employing Theorem 2.4 we deuce that  $L$  is a positive definite kernel. Now we use the strategy in [13] to construct the required Hilbert  $C^*$ -module  $\mathcal{F}$  and the map  $V$ .

For any map  $f : S \rightarrow \mathcal{E}$  with finite support we define  $\mathbb{L}f : S \rightarrow \mathcal{E}$  by

$$\mathbb{L}f(s) := \sum_{t \in S} L(s, t)f(t).$$

Denote by  $\mathcal{F}_0$  the semi-inner product  $\mathcal{A}$ -module of all maps  $\mathbb{L}f$  equipped with the pointwise operations and the semi-inner product

$$\langle \mathbb{L}f, \mathbb{L}g \rangle := \sum_{s, t \in S} \langle L(s, t)f(t), g(s) \rangle.$$

A standard argument based on the Cauchy–Schwarz inequality (1.1) shows that  $\langle \cdot, \cdot \rangle$  is indeed an inner product. We denote the completion of  $\mathcal{F}_0$  by  $\mathcal{F}$ , which is called the *reproducing kernel space*. It is easy to verify that  $\mathcal{L}(\mathcal{E}, \mathcal{F})$  is a Hilbert  $C^*$ -module over  $\mathcal{L}(E)$  via the inner product  $\langle T, S \rangle := T^*S$ .

Next, we set  $V : S \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{F})$  by  $V(s)(x) = \mathbb{L}(x_s)$ , where  $x_s : S \rightarrow \mathcal{E}$  is defined by

$$x_s(t) = \begin{cases} 0 & t \neq s \\ x & t = s \end{cases}.$$

It is not hard to see that  $V$  is well-defined. In addition,  $V(s)^*V(t) = L(s, t)$ , since

$$\langle V(s)^*V(t)x, y \rangle = \langle \mathbb{L}x_t, \mathbb{L}y_s \rangle = \langle L(s, t)x, y \rangle$$



for all  $x, y \in \mathcal{E}$ . It is notable that  $\cup_{s \in S} V(s)E$  is dense in  $\mathcal{F}$ . We have

$$\begin{aligned}
 & 2V(s)^*V(t) - V(s)^*V(s) - V(t)^*V(t) \\
 &= 2\left(\operatorname{Re}(V(s)^*V(t)) + 2i \operatorname{Im}(V(s)^*V(t))\right) - V(s)^*V(s) - V(t)^*V(t) \\
 &= 2i \operatorname{Im}L(s, t) + 2\operatorname{Re}L(s, t) - L(s, s) - L(t, t) \\
 &= 2i \operatorname{Im}L(s, t) + \operatorname{Re}K(s, t) - \frac{1}{2}(K(s, s) + K(t, t)) \\
 &= i \operatorname{Im}(K(s, t) - K(s, s_0) - K(s_0, t)) + \operatorname{Re}K(s, t) - \frac{1}{2}(K(s, s) + K(t, t)) \\
 &\quad \text{(since } K(s_0, s_0) \text{ is self-adjoint)} \\
 &= K(s, t) + \left(\frac{-1}{2}K(s, s) - i \operatorname{Im}K(s, s_0)\right) + \left(\frac{-1}{2}K(t, t) - i \operatorname{Im}K(t, s_0)\right)^* \\
 &= K(s, t) + h(s) + h(t)^*,
 \end{aligned}$$

where

$$h(s) := \frac{-1}{2}K(s, s) - i \operatorname{Im}K(s, s_0). \quad (3.3)$$

Thus

$$K(s, t) = 2V(s)^*V(t) - V(s)^*V(s) - V(t)^*V(t) - h(s) - h(t)^*.$$

□

The triple  $(V, \mathcal{F}, h)$  (or  $(V, \mathcal{F})$  when  $h = 0$ , resp.) is called the minimal Kolmogorov decomposition of the conditionally positive definite kernel  $K$ .

The next result is related to the positive definiteness of functions of the form  $\psi(s, t) := \varphi(s - t)$ , where  $\varphi$  is a real function on  $\mathbb{R}^d$ . It is a  $C^*$ -version of a known result in the Euclidean space  $\mathbb{R}^d$ ; see [14, Theorem A] and references therein. It can be deduced from Theorem 3.1 but we provide a direct proof for it.

**Corollary 3.2.** *A matrix in  $\mathbb{M}_n(\mathcal{A})$  with self-adjoint entries and with zero diagonal entries is conditionally positive if and only if it is a sum of matrices of the form  $-|a_i - a_j|^2$  with  $a_1, \dots, a_n \in \mathcal{A}$ .*

*Proof.* If  $a_1, \dots, a_n$  are entries of a  $C^*$ -algebra  $\mathcal{A}$ , then the matrix  $-|a_i - a_j|^2$  is conditionally positive in  $\mathbb{M}_n(\mathcal{A})$  with self-adjoint elements and with zero diagonal entries. To prove this, let us use Corollary 2.6 and the fact that

$$-|a_i - a_j|^2 = 2\langle a_i, a_j \rangle - |a_i|^2 - |a_j|^2.$$

Conversely, let  $A = [a_{ij}] \in \mathbb{M}_n(\mathcal{A})$  be a conditionally positive matrix with self-adjoint entries and with and with zero diagonal entries. Utilizing Corollary 2.6 we

get a positive matrix  $[b_{ij}] \in \mathbb{M}_n(\mathcal{A})$  and self-adjoint elements of  $c_1, \dots, c_n \in \mathcal{A}$  such that  $a_{ij} = b_{ij} + (c_i + c_j)$ . Hence  $[b_{i,j}] = \sum_{k=1}^m [d_i^{k*} d_j^k]$  for some  $d_i^k \in \mathcal{A}$  ( $1 \leq i \leq n, 1 \leq k \leq m$ ). From  $a_{ii} = 0$  ( $1 \leq i \leq n$ ) we conclude that  $2c_i = -b_{ii} = -\sum_{k=1}^m |d_i^k|^2$ . Since all  $a_{ij}$ 's and  $c_i$ 's are self-adjoint, we infer that so are all  $b_{ij}$ 's. Hence  $\sum_{k=1}^m d_i^{k*} d_j^k = b_{ij} = b_{ij}^* = b_{ji} = \sum_{k=1}^m d_j^{k*} d_i^k$ . Thus

$$a_{ij} = \sum_{k=1}^m d_i^{k*} d_j^k - \frac{1}{2} \sum_{k=1}^m |d_i^k|^2 - \frac{1}{2} \sum_{k=1}^m |d_j^k|^2 = - \sum_{k=1}^m \left| \frac{d_i^k}{\sqrt{2}} - \frac{d_j^k}{\sqrt{2}} \right|^2.$$

□

The problem of embedding of spaces endowed by types of metrics into Hilbert spaces goes back to the work of Schoenberg [17]. We aim to provide some conditions for embedding of a  $C^*$ -metric space into a Hilbert  $C^*$ -module. The next result has apparently an intrinsic relation to Corollary 3.2.

**Theorem 3.3.** *Let  $(S, d(\cdot, \cdot))$  be a  $C^*$ -metric space with values in a  $C^*$ -algebra  $\mathcal{A}$ . Then  $S$  is  $C^*$ -isometric to a subset of a Hilbert  $C^*$ -module if and only if  $K(s, t) = -d(s, t)^2$  is a conditionally positive definite kernel.*

*Proof.* Let  $V$  be a  $C^*$ -isometry from  $S$  into a Hilbert  $C^*$ -module  $(\mathcal{E}, \langle \cdot, \cdot \rangle)$  over a  $C^*$ -algebra  $\mathcal{A}$ . Put  $L : S \times S \rightarrow \mathcal{A}$  by  $L(s, t) = \langle V(s), V(t) \rangle$ . Then  $L$  is a positive definite kernel since for any  $x_1, \dots, x_n \in \mathcal{E}$  the Gram matrix  $[\langle x_i, x_j \rangle]$  is positive; cf. [10, Lemma 4.2]. In addition,

$$\begin{aligned} K(s, t) &= -d(s, t)^2 = -|V(s) - V(t)|^2 = -\langle V(s) - V(t), V(s) - V(t) \rangle \\ &= -L(s, s) - L(t, t) + 2\operatorname{Re}(L(s, t)) = P(s, t) + Q(s, t), \end{aligned}$$

where  $P(s, t) := -L(s, s) - L(t, t)$  is conditionally positive definite and  $Q(s, t) = 2\operatorname{Re}(L(s, t)) = L(s, t) + L(t, s)$  is positive definite. Hence  $K$  is clearly a conditionally positive definite kernel.

Conversely, let  $K(s, t) = -d(s, t)^2$  be conditionally positive definite. Fixing  $s_0 \in S$  and consider the positive definite kernel  $L(s, t) := \frac{1}{2}(K(s, t) - K(s, s_0) - K(s_0, t) + K(s_0, s_0))$ . Let us use the construction in Theorem 3.1 to get the Hilbert  $C^*$ -module  $\mathcal{L}(\mathcal{A}, \mathcal{F})$  and the map  $V : S \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{F})$  (Note that in Murphy's construction, we can start our work with a positive definite kernel with values in the set of 'compact' operators  $\mathcal{K}(\mathcal{E})$  acting on a Hilbert  $C^*$ -module  $\mathcal{E}$ . In our case, we deal with the Hilbert  $C^*$ -module  $\mathcal{A}$  over itself via the inner product

$\langle a, b \rangle = a^*b$  and apply the fact that  $\mathcal{K}(\mathcal{A})$  is nothing than  $\mathcal{A}$ . Then we have

$$\begin{aligned} |V(s) - V(t)|^2 &= (V(s) - V(t))^*(V(s) - V(t)) \\ &= L(s, s) + L(t, t) - 2\operatorname{Re}(L(s, t)) \\ &= -K(s, t) = d(s, t)^2. \end{aligned}$$

□

Finally we study an order on the space of conditionally positive definite kernels. We say  $K' \leq K$ , where  $K, K'$  are conditionally positive definite kernels whenever  $K - K'$  is conditionally positive definite. The next theorem provides a characterization of conditionally positive definite kernels majorized by a given kernel  $K$  under mild conditions. To get a suitable adjointable map, Pellonpää [16] considers a interesting regular condition. We say a kernel  $K : S \times S \rightarrow \mathcal{L}(\mathcal{E})$  is regular if the Hilbert  $C^*$ -module  $\mathcal{F}$  of its minimal Kolmogorov decomposition  $(V, \mathcal{F})$  is self-dual. Then, by [12, Proposition 2.5.2], every bounded  $A$ -linear map on  $\mathcal{F}$  is adjointable. In the case when  $\mathcal{A}$  is finite dimensional (or equivalently, is a direct sum of matrix algebras  $\mathbb{M}_m$ ), then every  $\mathcal{A}$ -module is self-dual.

**Theorem 3.4.** *Let  $K, K'$  be regular conditionally positive definite kernels on a set  $S$  into the  $C^*$ -algebra  $\mathcal{L}(\mathcal{E})$  for some Hilbert  $C^*$ -module  $\mathcal{E}$  such that the minimal Kolmogorov decompositions  $(V, \mathcal{F})$  of  $K$  is regular. Let there be  $s_0 \in S$  such that  $K(s, s_0)$  and  $K'(s, s_0)$  are self-adjoint for all  $s \in S$  and both  $K$  and  $K'$  vanish on the diagonal  $\{(s, s) : s \in S\}$ . Then  $K' \leq K$  if and only if there exists a positive contraction  $C \in \mathcal{L}(\mathcal{F})$  such that*

$$K'(s, t) = 2V(s)^*C^*CV(t) - V(s)^*C^*CV(s) - V(t)^*C^*CV(t)$$

for all  $s, t \in S$ .

*Proof.* We use construction and notation in Theorem 3.1. Let  $(V, \mathcal{F})$  and  $(V', \mathcal{F}')$  be the minimal Kolmogorov decompositions of  $K$  and  $K'$ , respectively. By the assumption,  $K(s, s) = 0$  for all  $s \in S$  and there is  $s_0 \in S$  such that  $K(s, s_0)$  is self-adjoint in  $\mathcal{A}$  for all  $s \in S$ , so that  $\operatorname{Im}K(s, s_0) = 0$  for all  $s \in S$ . Hence  $h(s) = 0$  for all  $s \in S$ . A similar assertion holds about the function  $h'$  corresponding to  $K'$ . Further,  $K' \leq K$  if and only if  $L' \leq L$  where  $L$  and  $L'$  are corresponding positive definite kernels to  $K$  and  $K'$  according to (3.2), respectively. By the Kolmogorov construction,  $L' \leq L$  if and only if  $\langle L'f, L'f \rangle \leq \langle Lf, Lf \rangle$  for all  $f : S \rightarrow E$  with finite support. Now we show that this latter inequality is

equivalent to  $V'(s) = CV(s)$  for some positive contraction  $C \in \mathcal{L}(\mathcal{F})$ :

Let  $\langle \mathbb{L}'f, \mathbb{L}'f \rangle \leq \langle \mathbb{L}f, \mathbb{L}f \rangle$  for all  $f : S \rightarrow E$  with finite support. Then the  $\mathcal{A}$ -linear map  $W : \mathcal{F}_0 \rightarrow \mathcal{F}'_0$  defined by  $W(\mathbb{L}f) = \mathbb{L}'f$  is a contraction and can be extended to a contraction, denoted by the same  $W$ , from  $\mathcal{F}$  to  $\mathcal{F}'$ . It follows from [12, Proposition 2.5.2] that  $W$  is self-adjoint. Thus for all  $x \in \mathcal{E}$ ,

$$V'(s)(x) = \mathbb{L}'(x_s) = W(\mathbb{L}(x_s)) = (WV(s))(x),$$

whence  $V'(s) = WV(s)$ . Thus

$$L'(s, t) = V'(s)^*V'(t) = (WV(s))^*(WV(t)) = V(s)^*W^*WV(t) = V(s)^*CV(t),$$

where  $C = W^*W$  and  $\|C\| \leq \|W\|^2 \leq 1$  since  $W$  is a contraction.

Conversely, let  $V'(s) = CV(s)$  for some positive contraction  $C \in \mathcal{L}(\mathcal{F})$ . Then

$$\begin{aligned} \langle \mathbb{L}'f, \mathbb{L}'f \rangle &= \sum_{s, t \in S} \langle K'(s, t)f(t), f(s) \rangle \\ &= \sum_{s, t \in S} \langle V'(s)^*V'(t)f(t), f(s) \rangle \\ &= \sum_{s, t \in S} \langle V(s)^*C^*CV(t)f(t), f(s) \rangle \\ &= \langle C \sum_{t \in S} V(t)f(t), C \sum_{s \in S} V(s)f(s) \rangle \\ &= \left\| C \sum_{s \in S} V(s)f(s) \right\|^2 \\ &\leq \left\| \sum_{s \in S} V(s)f(s) \right\|^2 \\ &= \sum_{s, t \in S} \langle K(s, t)f(t), f(s) \rangle \\ &= \langle \mathbb{L}f, \mathbb{L}f \rangle \end{aligned}$$

Finally, by employing (3.1), we observe that  $K' \leq K$  if and only if

$$\begin{aligned} K'(s, t) &= 2V'(s)^*V'(t) - V'(s)^*V'(s) - V'(t)^*V'(t) \\ &= 2V(s)^*C^*CV(t) - V(s)^*C^*CV(s) - V(t)^*C^*CV(t). \end{aligned}$$

□

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